数理解析研究所講究録
第 1936 巻 2015 年 120-131

Towards the variation of Jorgensen's theory for the torus with a single cone point*

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1 Introduction

In his famous unfinished paper [6], Jorgensen gave a description of the combinatorial structure of the Ford domain of a once-punctured torus Kleinian group. As pointed out by Sullivan [9], there seems to be a parallel theory if we replace the "puncture" to a "cone singularity". In fact, Jorgensen [7] gave examples of doubly degenerate groups with cone angle $2\pi/n$ for natural numbers n, and applied them to construct hyperbolic structures for certain closed surface bundles over the circle. In this article, I will give an overview of the project to establish a variation of Jorgensen's theory for the cone manifolds obtained from the original once-punctured torus by replacing the puncture to a single cone point of cone angle $\theta \in (0, 2\pi)$.

2 Torus with a single cone point

Let θ be a real number with $0 < \theta < 2\pi$. Let T be the torus and v a point in T. We denote the triplet $(T, \{v\}, \theta)$ by T_θ and call it the torus with a single cone point v with cone angle θ. Set $M = T \times \mathbb{R}$ and $\Sigma = \{v\} \times \mathbb{R} \subset M$, and denote the triplet (M, Σ, θ) by M_θ (see Figure 1).

Let S_θ be the intersection of two half spaces of $\mathbb{H}^3$ with dihedral angle θ at the intersection ℓ of the boundary planes, and $\mathbb{H}^3_\theta$ the quotient space obtained from S_θ by identifying the pairs of points in ∂S_θ by the rotation about ℓ of angle θ (see Figure 2). A *standard ball of angle* θ is defined to be a ball in $\mathbb{H}^3_\theta$ centered at a point in the image of ℓ, and a *standard horoball of angle* θ is defined to be the projected image in $\mathbb{H}^3_\theta$ of the intersection of S_θ and a horoball centered at an endpoint of ℓ.

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*This work was supported by JSPS KAKENHI Grant Number 23740064.

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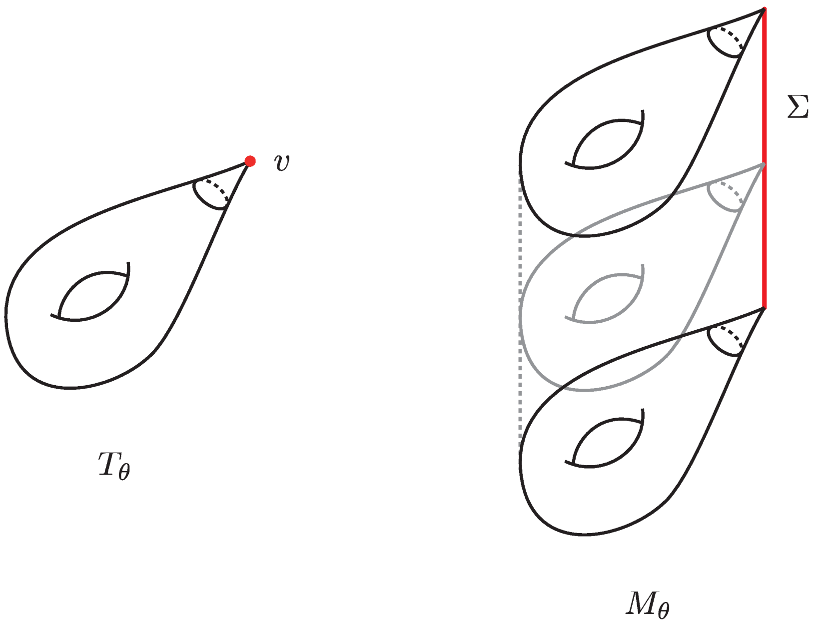


Figure 1: The cone manifolds T_θ and M_θ

A cone hyperbolic structure on M_θ is a length metric on M_θ such that (i) each point in $M - \Sigma$ has a neighborhood isometric to a ball in $\mathbb{H}^3$, and (ii) each point in Σ has a neighborhood isometric to a standard ball of angle θ.

Set $T_0 = T - \{v\}$ and $M_0 = M - \Sigma$. Then the projection $M_0 \to T_0 \times \{0\} \approx T_0$ induces the isomorphism $\pi_1(M_0) \cong \pi_1(T_0)$; we denote the group by G. We fix a peripheral loop in T_0 and denote it by κ (see Figure 3). Associated with a cone hyperbolic structure on M_θ, we obtain a smooth incomplete hyperbolic structure on M_0, and hence the holonomy representation $\rho : G \to PSL(2, \mathbb{C})$. For a holonomy representation ρ, we have $\operatorname{tr} \rho(\kappa) = \pm 2 \cos(\theta/2)$.

3 Space of representations

3.1 Elliptic generators

When we study the space of representations of G into $PSL(2, \mathbb{C})$ or $SL(2, \mathbb{C})$, it is convenient to work with the orbifold fundamental group $\widehat{G}$ of the orbifold $\mathcal{O}_0 = (S^2; \infty, 2, 2, 2)$, the orbifold with the once-punctured sphere as underlying space and with three singular points of order 2, obtained as the quotient of T_0 by the elliptic involution. Denote the canonical projection by $\mathrm{pr}_F : T_0 \to \mathcal{O}_0$. The group $\widehat{G}$ has a presentation

$$\widehat{G} = \langle P_0, Q_0, R_0 \,|\, P_0^2, Q_0^2, R_0^2 \rangle,$$

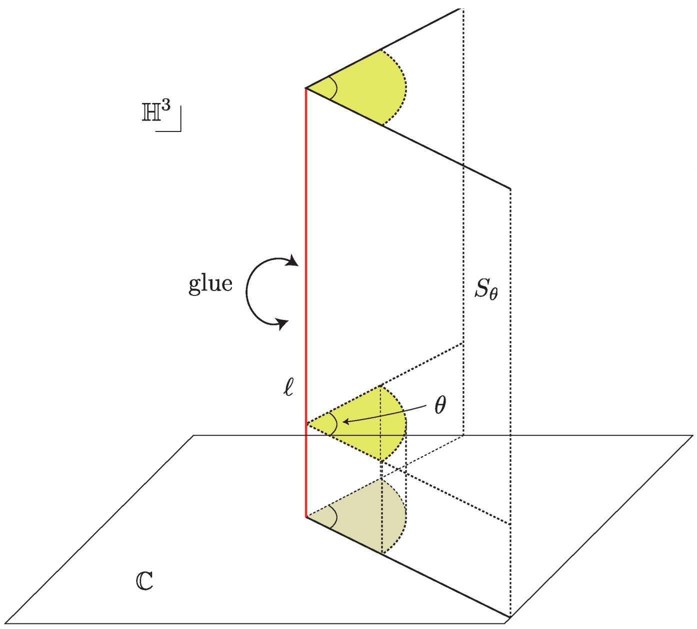


Figure 2: Neighborhood of a point in the cone singularity

where each P_0, Q_0 and R_0 is represented by a loop which encircles a singular point, and $K = R_0Q_0P_0$ is represented by a peripheral loop of $\mathcal{O}_0$ such that $\mathrm{pr}_{F*}(\kappa) = K^2$. An *elliptic generator triple* is a triple (P, Q, R) of elements of order 2 in $\widehat{G}$ such that $\widehat{G} = \langle P, Q, R\rangle$ and $RQP = K$. Each P, Q and R in an elliptic generator triple is called an *elliptic generator*. For any elliptic generator P, the element KP is contained in $\mathrm{pr}_{F*}(G)$ and represented by a simple loop in T_0 obtained as the image of a straight line in the universal abelian cover $\mathbb{R}^2 - \mathbb{Z}^2$ whose slope is a rational number or ∞. We call the slope of the straight line the *slope of* P and denote by $s(P)$. Let $\mathcal{D}$ be the Farey complex, namely, $\mathcal{D}$ is the 2-dimensional simplicial complex embedded in $\overline{\mathbb{H}^2}$ such that the set of 2-simplices is $\{\gamma\langle\infty, 0, 1\rangle \mid \gamma \in PSL(2, \mathbb{Z})\}$, where $\partial\mathbb{H}^2$ is identified with $\widehat{\mathbb{R}} = \mathbb{R} \cup \{0\}$, and $\langle\infty, 0, 1\rangle$ denotes the ideal triangle with vertices ∞, 0 and 1 (see Figure 3). The set of vertices of $\mathcal{D}$ is equal to $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{0\}$. The following property is well-known (see [2, Section 2.1] for example):

1. If (P, Q, R) is an elliptic generator triple, then any consecutive three elements in the following sequence is also an elliptic generator triple:
$$\ldots, R^{K^{-2}}, P^{K^{-1}}, Q^{K^{-1}}, R^{K^{-1}}, P, Q, R, P^K, Q^K, R^K, P^{K^2}, \ldots$$
Here X^Y denotes the conjugate YXY^{-1}.

2. If (P, Q, R) is an elliptic generator triple, then so are (P, R, Q^R) and (Q^P, P, R).

3. Any elliptic generator triple is obtained from (P_0, Q_0, R_0) by a finite sequence of operations in 1 and 2.

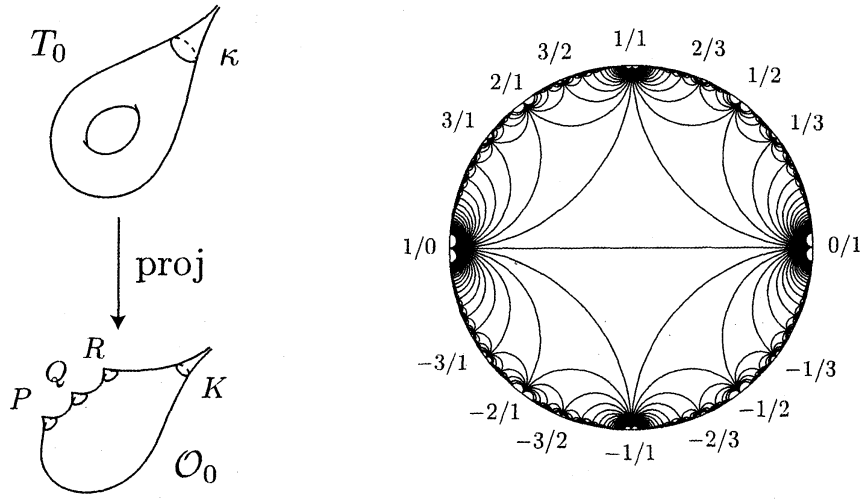


Figure 3: Punctured torus and the quotient orbifold, and the Farey complex $\mathcal{D}$

4. For any elliptic generator triple (P, Q, R), $\sigma = \langle s(P), s(Q), s(R)\rangle$ is a triangle in $\mathcal{D}$, which is invariant under the operation of 1. The sequence in 1 is called the *sequence of elliptic generators associated with* σ.

3.2 Space of representations containing holonomy representations

As mentioned in Section 2, the holonomy representation of a cone hyperbolic structure on M_θ induces the holonomy representation $\rho : G \to PSL(2,\mathbb{C})$ such that $\operatorname{tr}\rho(\kappa) = \pm 2\cos(\theta/2)$. We call a representation of a group into $SL(2,\mathbb{C})$ or $PSL(2,\mathbb{C})$ to be *elementary* if the image has a fixed point in $\overline{\mathbb{H}^3}$. We introduce the following representation spaces, where the relation $\sim$ is induced from the conjugacy in the target group and we use the symbol $\mathrm{pr}_M : SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ for the projection:

- $\widetilde{\mathcal{R}}_\theta = \{\widetilde{\rho} : G \to SL(2,\mathbb{C}) :$ non-elementary $|\operatorname{tr}\widetilde{\rho}(K) = -2\cos(\theta/2)\}/\sim$
- $\mathcal{R}_\theta = \{\rho = \mathrm{pr}_M \circ \widetilde{\rho} : G \to PSL(2,\mathbb{C}) \,|\, \widetilde{\rho} \in \widetilde{\mathcal{R}}_\theta\}/\sim$
- $\widehat{\mathcal{R}}_\theta = \{\widehat{\rho} : \widehat{G} \to PSL(2,\mathbb{C}) :$ non-elementary $|\,\rho(K) = (\theta/2)$-rotation on $\mathbb{H}^3\}$

We also denote by Φ_θ the set of $(2 - 2\cos(\theta/2))$-Markoff maps in the sense of [10], namely, we set

$$\Phi_\theta = \{(x, y, z) \in \mathbb{C}^3 \,|\, x^2 + y^2 + z^2 - xyz - 2 = -2\cos(\theta/2)\}.$$

As in the case of once-punctured torus groups, there is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on $\widetilde{\mathcal{R}}_\theta$ which keeps invariant the representation in $\mathcal{R}_\theta$ obtained by the post-composition of pr_M.

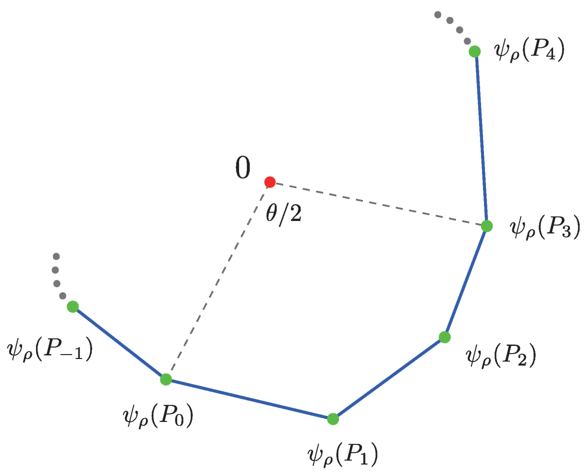


Figure 4: The values of ψ_ρ for a sequence of elliptic generators

This induces the 4-to-1 correspondence between $\widetilde{\mathcal{R}}_\theta$ and $\mathcal{R}_\theta$. We can see that the map $\widehat{\mathcal{R}}_\theta \to \mathcal{R}_\theta$ induced from the inclusion $G \to \widehat{G}$ is bijective. Also, there is a subset $\Phi_\theta^{\mathrm{ne}}$ of Φ_θ which is in 1-to-1 correspondence with $\widetilde{\mathcal{R}}_\theta$ by the theory of generalized Markoff maps [10]. These correspondence provides a framework parallel to that for once-punctured torus groups.

$$\begin{tikzcd} & \widetilde{\mathcal{R}}_\theta \arrow[r, leftrightarrow, "1:1"] \arrow[d, "4:1"] & \Phi_\theta^{\mathrm{ne}} \\ \widehat{\mathcal{R}}_\theta \arrow[r, leftrightarrow, "1:1"] & \mathcal{R}_\theta & \end{tikzcd}$$

3.3 Geometric parametrization

We can define a geometric parametrization for $\widehat{\mathcal{R}}_\theta$ which plays the counterpart of the *complex probability* introduced by Jorgensen in the theory of once-punctured torus groups. In what follows, we always use a representative for $\rho \in \widehat{\mathcal{R}}_\theta$ such that $\rho(K)$ maps each $z \in \mathbb{C}$ to $e^{i\theta/2}z$.

Let $\mathcal{EG}$ be the set of elliptic generators. To each $\rho \in \widehat{\mathcal{R}}_\theta$, we associate a map $\psi_\rho : \mathcal{EG} \to \widehat{\mathbb{C}}$ defined by $\psi_\rho(P) = \rho(P)(\infty)$. From the choice of representatives, this map is well-defined up to a multiple of a non-zero complex number. In fact, we have the following, and hence the map $\psi_\rho : \mathcal{EG} \to \mathbb{C}$ gives a parametrization for $\widehat{\mathcal{R}}_\theta$. (See Figure 4 which illustrates the values of ψ_ρ for a sequence of elliptic generators.)

Proposition 3.1. *For $\rho, \rho' \in \widehat{\mathcal{R}}_\theta$, $\rho = \rho'$ if and only if $\psi_\rho = \lambda\psi_{\rho'}$ for some $\lambda \in \mathbb{C} - \{0\}$.*

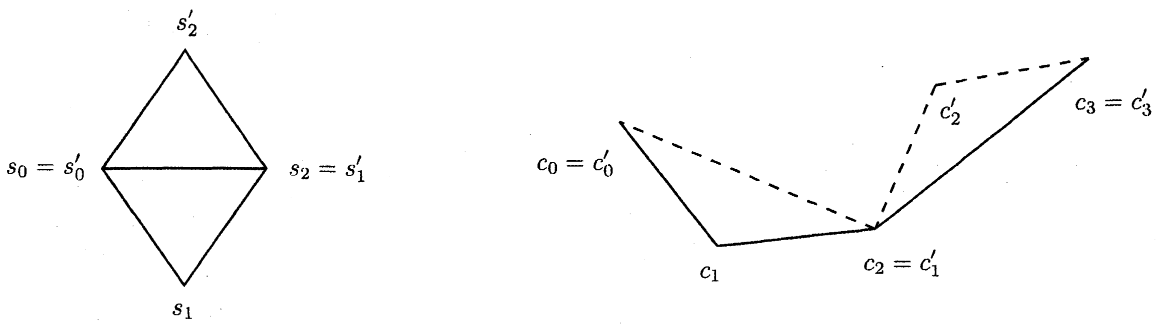


Figure 5: Switch of sequences of elliptic generators

Idea of Proof. First, suppose that $\rho = \rho'$, namely, there exists $T \in PSL(2,\mathbb{C})$ such that $\rho'(g) = T\rho(g)T^{-1}$ for any $g \in \widehat{G}$. Then we obtain $T(\infty) = \infty$ and $T(0) = 0$ by the assumption $0 < \theta < 2\pi$. Thus there exists $\lambda \in \mathbb{C} - \{0\}$ such that $T(z) = \lambda z$ for any $z \in \mathbb{C}$, and hence $\psi_\rho(P) = \lambda\psi_{\rho'}(P)$ for any $P \in \mathcal{EG}$. Next, suppose $\psi_\rho = \lambda\psi_{\rho'}$ for $\lambda \in \mathbb{C} - \{0\}$. By taking a suitable conjugate, we may assume that $\psi_\rho = \psi_{\rho'}$. We can show, by using the assumption that $0 < \theta < 2\pi$, that there is a sequence of elliptic generators $\{P_j\}$ such that $\psi_\rho(P_j) \neq \infty$ for any $j \in \mathbb{Z}$. From the property of a sequence of elliptic generators and the normalization of ρ and ρ', both $\rho(P_j)$ and $\rho'(P_j)$ enjoy the following same equation on $X \in PSL(2,\mathbb{C})$ for any $j \in \mathbb{Z}$:

$$X(\infty) = \psi_\rho(P_j), \quad X(\psi_\rho(P_{j-1})) = \psi_\rho(P_{j+1}), \quad X(\psi_\rho(P_{j+1})) = \psi_\rho(P_{j-1}).$$

Thus we obtain $\rho(P_j) = \rho'(P_j)$ for any $j \in \mathbb{Z}$. Since $\{P_j\}$ is a sequence of elliptic generators, this implies $\rho = \rho'$. □

The value of ψ_ρ for sequences of elliptic generators associated with adjacent triangles in $\mathcal{D}$ can be calculated by a method analogous to that for complex probabilities (see Figure 5). Let $\{P_j\}$ and $\{P_j'\}$ be sequences of elliptic generators with $P_0' = P_0$, $P_1' = P_2$ and $P_2' = P_2P_1P_2$. Then these sequences are associated with a pair of adjacent triangles in $\mathcal{D}$. Let $\rho \in \widehat{\mathcal{R}}_\theta$ such that none of $c_j = \psi_\rho(P_j)$ and $c_j' = \psi_\rho(P_j')$ for $j \in \mathbb{Z}$ is equal to ∞. Then the sequence $\{c_j'\}$ is obtained from $\{c_j\}$ as follows. Let $j = 3k + l$ for $k \in \mathbb{Z}$ and $l \in \{0,1,2\}$. If $l = 0$ (resp. $j = 1$), then $P_j' = P_j$ (resp. $P_j' = P_{j+1}$), and hence $c_j' = c_j$ (resp. $c_j' = c_{j+1}$). If $l = 2$, then there is a orientation-preserving similarity transformation of $\mathbb{C}$ which maps the three points c_{j-2}, c_{j-1} and c_j to c_{j+1}', c_j' and c_{j-1}', respectively. This characterizes $\{c_j'\}$.

4 Good fundamental polyhedron

Let $\rho \in \widehat{\mathcal{R}}_\theta$. In order to define a *good fundamental polyhedron* for ρ, we introduce several conditions analogous to those for once-punctured torus groups (cf. [2]). Following [2], we denote by $I(\gamma)$ (resp. $Ih(\gamma)$) the isometric circle (resp. the isometric hemisphere) for $\gamma \in PSL(2,\mathbb{C})$ with $\gamma(\infty) \neq \infty$. We also denote the inside (resp. outside) of $I(\gamma)$ by $D(\gamma)$ (resp. $E(\gamma)$), and the inside (resp. outside) of $Ih(\gamma)$ by $Dh(\gamma)$ (resp. $Eh(\gamma)$).

Let $\{P_j\}$ be a sequence of elliptic generators such that $\psi_\rho(P_j) \neq \infty$ for any $j \in \mathbb{Z}$. For each $j \in \mathbb{Z}$, set $c_j = \psi_\rho(P_j)$ and denote the segment in $\mathbb{C}$ with endpoints c_j and c_{j+1} by l_j, and suppose that l_j does not contain the origin. Let $l : \mathbb{R} \to \mathbb{C} - \{0\}$ be the map such that the restriction to the closed interval $[j, j+1]$ is the affine map into $\mathbb{C}$ satisfying $l(j) = c_j$ and $l(j+1) = c_{j+1}$. Then we have $l(t+3k) = e^{ik\theta/2}l(t)$ for any $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. (See Figure 4.)

Let $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ be the universal covering, and let $\widetilde{d}$ be the metric on $\mathbb{C}$ obtained as the pull-back of the Euclidean metric on $\mathbb{C} - \{0\}$ by the covering map exp. We denote the metric space $(\mathbb{C}, \widetilde{d})$ by $\widehat{\mathbb{C}}_0$. Let $\widetilde{l} : \mathbb{R} \to \widehat{\mathbb{C}}_0$ be a continuous lift of l by exp. We define the isometric action of the infinite cyclic group $\mathbb{Z}$ on $\mathbb{R}$ (resp. $\widehat{\mathbb{C}}_0$) by $1 \cdot t = t + 3$ (resp. $1 \cdot z = z + i\theta/2$). Then $\widetilde{l}$ is equivariant with respect to these actions of $\mathbb{Z}$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $\mathbb{C}_\theta = \widehat{\mathbb{C}}_0/\mathbb{Z}$ equipped with the metrics so that the covering projections are local isometries. We remark that $\mathbb{C}_\theta$ can be naturally regarded as the "boundary" of the model space $\mathbb{H}^3_\theta$. We denote $\mathbb{H}^3_\theta \cup \mathbb{C}_\theta$ by $\overline{\mathbb{H}^3_\theta}$. Then $\widetilde{l}$ induces the map $l_\theta : S^1 \to \mathbb{C}_\theta$ whose image is the union of three (geodesic) segments $l_\theta([j, j+1])$ $(j \in \{0, 1, 2\})$. We denote the image of l_θ in $\mathbb{C}_\theta$ by $\mathcal{L}_\theta(\rho, \sigma)$. Under the above notation, we say that ρ satisfies the condition *Simple* at σ if $l_\theta : S^1 \to \mathbb{C}_\theta$ is a homeomorphism onto its image $\mathcal{L}_\theta(\rho, \sigma)$ and also $\mathcal{L}_\theta(\rho, \sigma)$ bounds the bounded (resp. unbounded) component of $\mathbb{C}_\theta - \mathcal{L}_\theta(\rho, \sigma)$ in its left (reap. right) hand side.

For $\rho \in \mathcal{R}_\theta$ which satisfies the condition Simple at σ, let ξ_j be the length of l_j for each $j \in \mathbb{Z}$. By definition, ξ_j is also equal to the length of the segment obtained as the image $l_\theta([j, j+1])$. We say ρ satisfies *the triangle inequality at* σ if $\sqrt{\xi_0}, \sqrt{\xi_1}, \sqrt{\xi_2}$ satisfies the triangle inequality. By a parallel argument to the case of once-punctured torus, ρ satisfies the triangle inequality at σ if and only if $I(\rho(P_j)) \cap I(\rho(P_{j+1}))$ consists of exactly two points for any $j \in \mathbb{Z}$.

We say that ρ is *admissible* at σ if ρ satisfies the condition Simple and triangle inequality at σ, and also if $D(\rho(P_j))$ does not contain the origin for any $j \in \mathbb{Z}$. The final condition corresponds to the condition NonZero introduced in [2] for the case of once-punctured torus. For ρ which is admissible at σ, we can define the *side parameter* $\boldsymbol{\theta}(\rho, \sigma) = (\theta^-(\rho), \theta^+(\rho))$ by a similar way to the case of once-punctured torus.

Let $\boldsymbol{\nu} = (\nu^-, \nu^+)$ be a pair of points in $\mathbb{H}^2$, and ℓ the geodesic segment in $\mathbb{H}^2$ with endpoints $\nu^\pm$. Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be the triangles in $\mathcal{D}$ such that the interior of σ_k intersects ℓ in this order, and denote the sequence $\{\sigma_1, \ldots, \sigma_m\}$ by $\Sigma(\boldsymbol{\nu})$, which is called a chain of triangles in [2]. We also define the 2-dimensional simplicial complex $\mathcal{L}(\boldsymbol{\nu}) = \mathcal{L}(\Sigma(\boldsymbol{\nu}))$ associated with $\boldsymbol{\nu}$ following [2]. As the argument in the above, where we define the condition Simple, there is a natural action of $\mathbb{Z}$

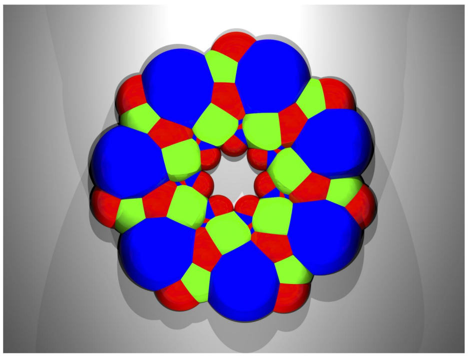


Figure 6: A developed image of a good fundamental polyhedron in $\mathbb{H}^3$

on $\mathcal{L}(\boldsymbol{\nu})$. We denote the quotient $\mathcal{L}(\boldsymbol{\nu})/\mathbb{Z}$ by $\mathcal{L}_\theta(\boldsymbol{\nu})$. We say that a pair $(\rho, \boldsymbol{\nu})$, which is called a *labeled representation*, satisfies the condition *Simple* if ρ satisfies the condition Simple at each σ_k, and if there is a linear extension of $S^1 \to \mathcal{L}_\theta(\rho, \sigma_k)$ $(k \in \{1, \ldots, m\})$ to $\mathcal{L}_\theta(\boldsymbol{\nu}) \to \mathbb{C}_\theta$ which is a homeomorphism onto the image.

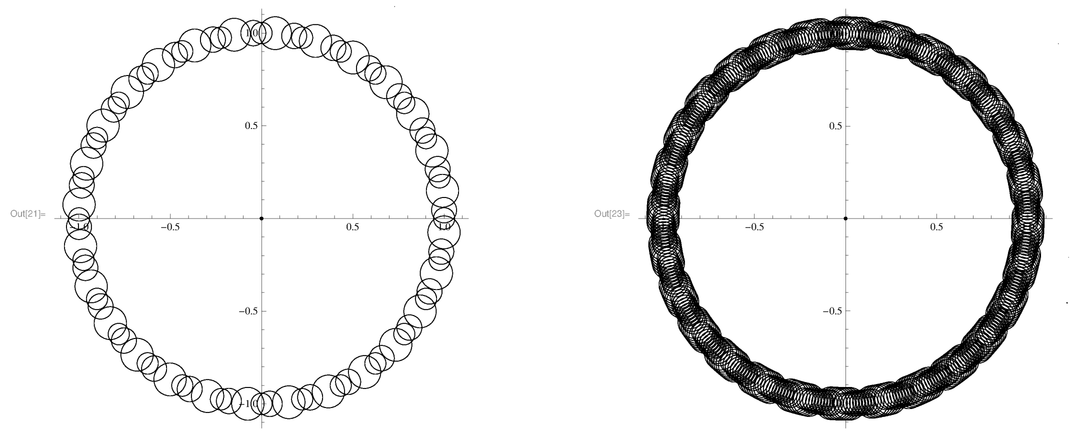
Let $(\rho, \boldsymbol{\nu})$ be a labeled representation which satisfies the condition Simple. Then we can define a "polyhedron" $Eh(\rho, \boldsymbol{\nu})$ in $\overline{\mathbb{H}^3_\theta}$ as the "common exterior" to the family of isometric hemispheres $\{Ih(\rho(P)) \mid s(P) \in \mathcal{L}(\boldsymbol{\nu})^{(0)}\}$. This definition is a slight modification of that is mentioned in Section 6.4 of [2], where a fundamental domain modulo the action of the peripheral subgroup is discussed. For the polyhedron $Eh(\rho, \boldsymbol{\nu})$, we define the two conditions *Duality* and *Frontier* by simply following Definitions 6.1.3 and 6.1.4 in [2].

A labeled representation $(\rho, \boldsymbol{\nu})$ is said to be *good* if it satisfies the condition Simple, and if the polyhedron $Eh(\rho, \boldsymbol{\nu})$ satisfies the conditions Duality and Frontier. We call $Eh(\rho, \boldsymbol{\nu})$ a *good fundamental polyhedron* for ρ. By following [2], we can see that a good fundamental polyhedron induces a complete cone hyperbolic structure on M_θ. See Figure 6, which illustrates a developed image of a good fundamental polyhedron for the cone angle $\theta = 4\pi/7$. We remark that the developed image does not make sense if the cone angle θ is an irrational multiple of π (see Figure 7).

By comparing the numerical results done by Yamashita based on his joint work with Tan [12] and by the author, we proposed the following conjecture.

Conjecture 4.1 (Akiyoshi-Yamashita). *For $\rho \in \widehat{\mathcal{R}}_\theta$, ρ has a good fundamental polyhedron if and only if ρ satisfies the BQ-condition.*

Except for the real representations described in the next section, this conjecture is still open.



θ: rational with π $\qquad$ θ: irrational with π

Figure 7: Rationality of cone angle with π and developed image

5 Real representations

In this section, we see a partial affirmative answer to Conjecture 4.1. To this end, we introduce the real slices of the representation spaces. Let $\widetilde{\mathcal{R}}^{\mathbb{R}}_{\theta}$ be the subspace of $\widetilde{\mathcal{R}}_{\theta}$ consisting of the representations with $SL(2,\mathbb{R})$-representations as representatives. Let $\mathcal{R}^{\mathbb{R}}_{\theta}$ be the subspace of $\mathcal{R}_{\theta}$ consisting of the representations $\mathrm{pr}_M \circ \widetilde{\rho}$ for $\widetilde{\rho} \in \widetilde{\mathcal{R}}^{\mathbb{R}}_{\theta}$, and let $\widehat{\mathcal{R}}^{\mathbb{R}}_{\theta}$ be the subspace of $\widehat{\mathcal{R}}_{\theta}$ corresponding to $\mathcal{R}^{\mathbb{R}}_{\theta}$. Goldman and Tan-Wong-Zhang made intensive studies on the space $\widetilde{\mathcal{R}}^{\mathbb{R}}_{\theta}$, in which they showed the following theorems.

Theorem 5.1 (Goldman [8]). *For $\rho \in \mathcal{R}^{\mathbb{R}}_{\theta}$, either (i) ρ is realized as the holonomy representation of a cone hyperbolic structure on T_θ, or (ii) ρ is elementary.*

Theorem 5.2 (Tan-Wong-Zhang [10]). *For $\rho \in \mathcal{R}^{\mathbb{R}}_{\theta}$, ρ satisfies the BQ-condition if and only if ρ is realized as the holonomy representation of a cone hyperbolic structure on T_θ.*

The following is the main theorem of [1].

Theorem 5.3. *Any non-elementary $\rho \in \mathcal{R}^{\mathbb{R}}_{\theta}$ has a good fundamental polyhedron.*

Summarizing the above three theorems, we obtain a partial affirmative answer to Conjecture 4.1 on $\mathcal{R}^{\mathbb{R}}_{\theta}$.

The proof of Theorem 5.3 uses a specialization of the geometric parameterization for $\widehat{\mathcal{R}}_{\theta}$ to $\widehat{\mathcal{R}}^{\mathbb{R}}_{\theta}$, which enables us to simplify the condition of good fundamental polyhedra to a certain algebraic condition for the parameter. Then we obtain the theorem by following the argument of Bowditch [4] and using the results of [10].

6 Uniqueness of a good fundamental polyhedron

In this section, we observe that a good fundamental polyhedron has a property similar to that for the Ford domain of a Kleinian group. In what follows, we suppose that M_θ is equipped with the complete hyperbolic structure induced from a good fundamental polyhedron Eh. Then there is a horoball $\widetilde{H}$ centered at ∞ such that the intersection $\widetilde{H} \cap Eh$ projects onto the subset H of M_θ isometric to a standard horoball with cone angle θ. Let C be the subset of M_θ obtained as the image of ∂Eh.

Let $x \in M_\theta - H$. Then the closed r-neighborhood $B(x, r)$ of x in M_θ intersects H for a sufficiently large positive number r. Since M_θ is a complete length space which is locally compact, $B(x, r)$ is compact by the Hopf-Rinow theorem. Thus there is an arc γ in $B(x, r)$ which connects x to ∂H such that the length of γ is equal to the distance between x and H. From the minimality of the length of γ, we see that either (i) γ is contained entirely in Σ, or (ii) γ is a geodesic disjoint from Σ which intersects ∂H perpendicularly at an endpoint. Then C is characterized as the cut locus of M_θ with respect to H, namely, the following holds. For any $\widetilde{x} \in Eh$, let $\widetilde{\gamma}_{\widetilde{x}}$ be the vertical geodesic segment in Eh connecting $\widetilde{x}$ to $\partial \widetilde{H}$, and $\gamma_{\widetilde{x}}$ be the projected image of $\widetilde{\gamma}_{\widetilde{x}}$ in M_θ.

Proposition 6.1. *Let $\widetilde{x} \in Eh$ which project onto $x \in M_\theta - H$.*

1. *Suppose that $\widetilde{x}$ is a point in the interior of Eh. Then $\gamma_{\widetilde{x}}$ is the unique shortest arc in M_θ connecting x to H.*

2. *Suppose that $\widetilde{x}$ is a point in ∂Eh. Let $\widetilde{x}_1, \dots, \widetilde{x}_k$ be the points in ∂Eh which project onto x, where $k \in \{2, 3, 4\}$. Then $\gamma_{\widetilde{x}_1}, \dots, \gamma_{\widetilde{x}_k}$ is the complete list of shortest arcs in M_θ connecting x to H.*

Idea of proof. We give the idea of the proof for the assertion 1. The assertion 2 can be proved by a similar argument. In the proof, we use the following property of good fundamental polyhedra:

(i) The boundary of a good fundamental polyhedron Eh is a union of isometric hemispheres.

(ii) For any point $x \in Eh$, there are at most three more points in Eh which are identified with x by the side pairings.

(iii) The points in ∂Eh that are identified by the side pairings have the same height in the upper half space model.

Let x be a point in $M_\theta - H$ such that there is a point $\widetilde{x}$ in the interior of Eh projecting onto x. Suppose to the contrary that there is a path δ distinct from $\gamma_{\widetilde{x}}$

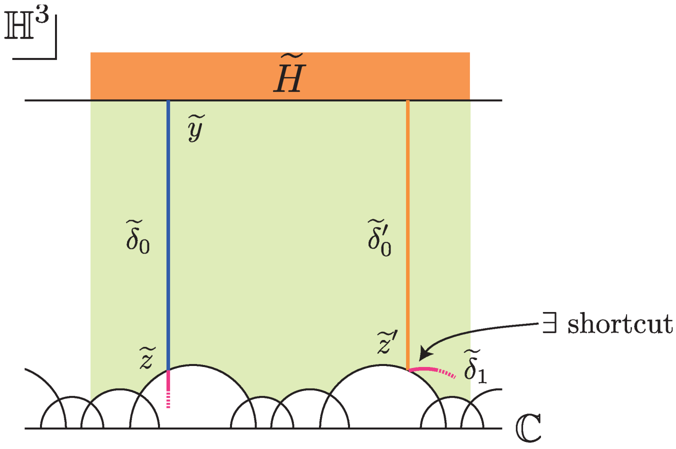


Figure 8: Shortcut between $\widetilde{\delta}'_0$ and $\widetilde{\delta}_1$ exists.

such that the length of δ is less than or equal to that of $\gamma_{\widetilde{x}}$. We may suppose that the length of δ is equal to the distance between x and H, and so δ is a geodesic which intersects ∂H perpendicularly at an endpoint $y \in \partial H$. Let $\widetilde{y}$ be the unique lift of y contained in $\partial \widetilde{H} \cap Eh$, and $\widetilde{\delta}_0$ be the connected component of the lift of δ in Eh containing $\widetilde{y}$. Then we can see that $\widetilde{y}$ is not contained in $\widetilde{\gamma}_{\widetilde{x}}$, and $\widetilde{\delta}_0$ connects $\widetilde{y}$ to a point, $\widetilde{z}$, in ∂Eh, where it intersects a face of Eh transversely. Then there is a point $\widetilde{z}'$ in ∂Eh and the component, $\widetilde{\delta}_1$, of the lift of δ such that $\widetilde{z}$ and $\widetilde{z}'$ are identified by the side pairing and that $\widetilde{\delta}_1$ contains $\widetilde{z}'$ as an endpoint (see Figure 8).

Let $\widetilde{\delta}'_0$ be the vertical geodesic segment connecting $\widetilde{z}'$ to $\partial \widetilde{H}$. Since $\widetilde{z}$ and $\widetilde{z}'$ have the same heights, the lengths of $\widetilde{\delta}_0$ and $\widetilde{\delta}'_0$ are the same. Thus we can obtain an arc δ' which has the same length with δ by replacing $\widetilde{\delta}_0$ with $\widetilde{\delta}'_0$. However, there is a shortcut between $\widetilde{\delta}'_0$ and $\widetilde{\delta}_1$. This contradicts the assumption that δ is the shortest arc connecting x to H. $\square$

Remark 6.2. We can see that a good fundamental polyhedron is unique for the hyperbolic structure it induces. However, since we have not seen the uniqueness of the hyperbolic structures for a given representation, there is a possibility that two distinct good fundamental polyhedra induce the same holonomy representation.

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